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ON THE PRODUCT OF MOORE-PENROSE INVERSES C^* -ALGEBRAS

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ABSTRACT

Let A be a C^* -algebra. Assume that a, b and q are in A .we show that if $(pa)^* pa + 1 - a^\dagger a$ and $(aq)(aq)^* + 1 - aa^\dagger$ are invertible, then paq is Moore-Penrose invertible and we give a formula for the Moore-Penrose inverse of *paq*.

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INTRODUCTION

Many researchers have extended some results of the generalized inverse in the last two decades. Throughout this paper A will be C^* -algebra. An element $a \in A$ is *Moore-Penrose invertible* if there exists $b \in A$ so that

$$aba = a\,,\ bab = b,\ (ab)^* = ab \text{ and } (ba)^* = ba.$$

We will denote the Moore-Penrose inverse of a by $a^\dagger$ .if $a^\dagger$ exists, then it is unique. By the uniqueness, we have

$$\left(a^\dagger\right)^\dagger = a \text{ and } \left(a^\dagger\right)^* = \left(a^*\right)^\dagger.$$

An element x in A is called *idempotent* if $x^2 = x$. A *Projection* $p \in A$ satisfies $p = p^* = p^2$. Note that if $x \in A^\dagger$, then $xx^\dagger$ and $x^\dagger x$ are projections. In addition,

$$\left(xx^\dagger\right)^\dagger = xx^\dagger \quad \text{and} \quad \left(x^\dagger x\right)^\dagger = x^\dagger x.$$

If $a, b \in A$ are invertible, then $(ab)^{-1} = b^{-1}a^{-1}$. This cannot be extended to the Moore-Penrose inverse of ab . This question has been raised by Mahzoon[1], Mbekhta[2] *and* Bouldin[3] .

Alizadeh[4,5] *and* Khosravi[6] show that $(ab)^{\dagger} = b^{\dagger}a^{\dagger}$ holds in a C^{*} _ algebra iff $a^{\dagger}a$ commutes with bb^{*} and $bb^{\dagger}$ commutes with aa^{*}. After that Harte[7] and Izumino[8], extended this formula on Hilbert spaces for closed range operators. Also, in Koliha[9], there is a proof in case of rings with involution. In this paper, under some conditions, we present a purely algebraic proof for finding the Moore-Penrose inverse of paq and we give explicit formula for the Moore-Penrose inverse of paq. It should be noted that our proof is different from the proofs given before.

1 Main results

Theorem *1 Assue that A is a unital C^{*} _algebra and $a \in A^{\dagger}$* with Moore-Penrose *Inverse Suppose that $p, q \in A$.* If $(pa)^{*}pa + 1 - a^{\dagger}a$ and $(aq)(aq)^{*} + 1 - aa^{\dagger}$ *are invertible, then paq is Moore-Penrose invertible and*

$$(paq)^{\dagger} = (aq)^{*}\left[aq(aq)^{*} + 1 - aa^{\dagger}\right]^{-1} a\left[(pq)^{*}pq + 1 - a^{\dagger}a\right]^{-1}(pa)^{*}.$$

Proof .For simplicity, let $t = paq$, $u = aq(aq)^{*} + 1 - aa^{\dagger}$ and $v = (pq)^{*}pq + 1 - a^{\dagger}a$. Since $a = aa^{\dagger}a$,thus $aq(aq)^{*} = aa^{\dagger}\left(aq(aq)^{*}\right)$.Note that $aa^{\dagger}$ and $a^{\dagger}a$ are projections. Then

(1)

$$\begin{aligned} aq(aq)^{*}u^{-1} &= \left[aa^{\dagger}aq(aq)^{*} - \left(aa^{\dagger}\right)^{2} + aa^{\dagger}\right]u^{-1} \\ &= aa^{\dagger}\left[aq(aq)^{*} + 1 - aa^{\dagger}\right]u^{-1} \\ &= aa^{\dagger}uu^{-1} = aa^{\dagger}. \end{aligned}$$

In a similar manner we obtain (2)

$$\begin{aligned} v^{-1}(pa)^{*}pa &= v^{-1}\left[(pa)^{*}paa^{\dagger}a\right] \\ &= v^{-1}\left[(pa)^{*}paa^{\dagger}a + a^{\dagger}a - \left(a^{\dagger}a\right)^{2}\right] \\ &= v^{-1}\left[(pa)^{*}pa + 1 - a^{\dagger}a\right]a^{\dagger}a = a^{\dagger}a. \end{aligned}$$

Now we set $x = (aq)^{*}u^{-1}av^{-1}(pa)^{*}$. Using (1)and (2)we get

$$\begin{aligned} txt &= paq\left[(aq)^{*}u^{-1}a\left(v^{-1}\left[(pa)^{*}pa\right]\right)q\right] \quad (3) \\ &= paq\left[(aq)^{*}u^{-1}aa^{\dagger}aq\right] \\ &= p\left(aq(aq)^{*}u^{-1}aq\right) \end{aligned}$$

$$= p\left(aa^{\dagger}a\right)q = paq = t,$$

$$xtx = (xt)x = \left[(aq)^{*} u^{-1} a\left(v^{-1}(pa)^{*} pa\right)q\right]x \quad (4)$$

$$= \left[(aq)^{*} uaa^{\dagger}aq\right]\left((aq)^{*} u^{-1} av^{-1}(pa)^{*}\right)$$

$$= (aq)^{*} u\left[aq(aq)^{*} u^{-1}\right]av^{-1}(pa)^{*}$$

$$= (aq)^{*} u\left(a^{\dagger}a\right)av^{-1}(pa)^{*}$$

$$= (aq)^{*} uav^{-1}(pa)^{*} = x$$

and

$$tx = paqx = p\left[aq(aq)^{*} u^{-1}\right]av^{-1}(pa)^{*} \quad (5)$$

$$= paa^{\dagger}av^{-1}(pa)^{*} = (pa)v^{-1}(pa)^{*}$$

$$= \left[(pa)v^{-1}(pa)^{*}\right]^{*} = (tx)^{*}.$$

In a similar manner one can get

$$(xt)^{*} = xt$$

From (3),(4),(5)and(6) we conclude that x which complete the proof.

An easy consequence of the above theorem is:

Corollary 2

Under the condition of the theorem we have:

(i) $(pa)^{\dagger} = \left[(pa)^{*} pa + 1 - a^{\dagger}a\right]^{-1}(pa)^{*};$

(ii) $(aq)^{\dagger} = (aq)^{*}\left[aq(aq)^{*} + 1 - aa^{\dagger}\right].$

CONCLUSION

In this paper we study the product of More-Penrose inverses in C*-algebras. If take he C*-algebra of matrices with the transpose conjugate, then it has a lot of applications in linear algebra.

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